

$$(M_1, g) \quad (M_2, h)$$

$$M_1 \times M_2 \quad f_1 \quad f_2 \quad M_1 \quad M_2$$

$$(f_2 M_1 \times_{f_1} M_2, G)$$

$$G = f_2^2 g + f_1^2 h$$

1969 Bishop

[1] 2016

[2] 2018

[3] 2022

Levi-Civita Ricci

[4]
5-6

[7] 1982 Michelsohn

[8]

1 0

[8] 2014

[9]

1985 Balas

[10]

Ricci

λ

λ

[10]

1987 Kobayashi

[11]

(M, J, G)

n

J

G

M

p

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M$$

J

$$\pm \sqrt{-1}.$$

$$z = (z^1, z^2, \dots, z^n)$$

$$T_p^{1,0} M \quad \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \dots, \frac{\partial}{\partial z^n} \right)$$

∇

G

J

[3]

$$\Gamma_{\gamma\alpha}^{\beta} = G^{\bar{\delta}\beta} \frac{\partial G_{\alpha\bar{\delta}}}{\partial z^{\gamma}} \quad 1$$

[12]

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$T_{\gamma\alpha}^{\beta} = \Gamma_{\gamma\alpha}^{\beta} - \Gamma_{\alpha\gamma}^{\beta} \quad 2$$

1 3

K [11]

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$K_{\alpha\gamma\bar{\delta}}^{\beta} = -\frac{\partial \Gamma_{\gamma\alpha}^{\beta}}{\partial \bar{z}^{\delta}} \quad 3$$

1 3

[11]

$$K_{\alpha}^{\beta} = G^{\bar{\delta}\gamma} K_{\alpha\gamma\bar{\delta}}^{\beta} \quad 4$$

[10]

(M, J, G)

$$T_{\gamma\alpha}^{\beta} = 0 \quad 5$$

(M, J, G)

[8]

(M, J, G)

1 0

$$\tau_{\gamma} = 0 \quad 6$$

(M, J, G)

$$\tau_{\gamma} = G^{\bar{\eta}\alpha} T_{\gamma\alpha\bar{\eta}} \quad 7$$

$$T_{\gamma\alpha\bar{\eta}} = G_{\beta\bar{\eta}} T_{\gamma\alpha}^{\beta} \quad 8$$

[11]

(M, J, G)

$$K = \varphi I, \quad i.e., \quad K_{\alpha}^{\beta} = \varphi \delta_{\alpha}^{\beta} \quad 9$$

 φ M

[13]

(M, J, G)

$$L = G^{\bar{\beta}\alpha} \frac{\partial^2}{\partial z^{\alpha} \partial \bar{z}^{\beta}} \quad 10$$

(M₁, g)(M₂, h)

m

n

M = M₁ × M₂

m + n

$$\pi_1: M \rightarrow M_1 \quad \pi_2: M \rightarrow M_2$$

$$z = (z_1, z_2) \in M, z_1 = (z^1, \dots, z^m) \in M_1$$

$$z_2 = (z^{m+1}, \dots, z^{m+n}) \in M_2 \quad \pi_1(z) = z_1 \quad \pi_2(z) = (z_2)$$

$$d\pi_1: T^{1,0}(M) \rightarrow T^{1,0}M_1, d\pi_2: T^{1,0}(M) \rightarrow T^{1,0}M_2 \quad \pi_1 \quad \pi_2$$

$$v = (v_1, v_2) \in T_z^{1,0}(M), v_1 = (v^1, \dots, v^m) \in T_{z_1}^{1,0}M_1, v_2 = (v^{m+1}, \dots, v^{m+n}) \in T_{z_2}^{1,0}M_2 \quad d\pi_1(z, v) = (z_1, v_1) \quad d\pi_2(z, v) = (z_2, v_2)$$

[3]

(M₁, g) (M₂, h)

$$f_1: M_1 \rightarrow (0, +\infty) \quad f_2: M_2 \rightarrow (0, +\infty)$$

(f₁M₁ ×_{f₁} M₂, G)

$$G: M \rightarrow (0, +\infty)$$

M = M₁ × M₂

$$G(z, v) = (f_2 \circ \pi_2)^2(z)g(\pi_1(z), d\pi_1(v)) + (f_1 \circ \pi_1)^2(z)h(\pi_2(z), d\pi_2(v)) \quad 11$$

$$z = (z_1, z_2) \in M, v = (v_1, v_2) \in T_z^{1,0}M \quad f_1 \quad f_2 \quad (M_1, g) \quad (M_2, h) \quad (f_1 M_1 \times_{f_1} M_2, G)$$

$$f_1 \equiv 1 \quad f_2 \equiv 1$$

(f₁M₁ ×_{f₁} M₂, G)(f₁M₁ ×_{f₁} M₂, G)f₁ f₂

$$f_1 \equiv 1 \quad f_2 \equiv 1$$

(f₁M₁ ×_{f₁} M₂, G)

$$1 \leq \alpha, \beta, \gamma, \eta, \delta \leq m+n, 1 \leq i, j, k, l, t \leq m, m+1 \leq i', j', k', l', t' \leq m+n. \quad (M_1, g) \quad (M_2, h)$$

$$1 \quad 2 \quad \Gamma_{jk}^1 \quad \Gamma_{j'k'}^2 \quad (M_1, g) \quad (M_2, h)$$

$$({}_{f_1}M_1 \times_{f_1} M_2, G)$$

$$(M_1, g) \quad (M_2, h)$$

$$g_{ij} = \frac{\partial^2 g}{\partial v^i \partial v^j}, \quad h_{i\bar{j}} = \frac{\partial^2 h}{\partial v^i \partial \bar{v}^j} \quad 12$$

G

[3]

$$(G_{\alpha\bar{\beta}}) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta} \right) = \begin{pmatrix} f_2^2 g_{ij} & 0 \\ 0 & f_1^2 h_{i\bar{j}} \end{pmatrix} \quad 13$$

$$(G^{\bar{\beta}\alpha}) \quad [3]$$

$$(G^{\bar{\beta}\alpha}) = \begin{pmatrix} f_2^{-2} g^{\bar{i}i} & 0 \\ 0 & f_1^{-2} h^{\bar{i}i'} \end{pmatrix} \quad 14$$

$$({}_{f_2}M_1 \times_{f_1} M_2, G)$$

$$[3] \quad ({}_{f_1}M_1 \times_{f_1} M_2, G)$$

$$\Gamma_{\gamma\alpha}^\beta$$

$$\Gamma_{jk}^i = \Gamma_{jk}^1, \quad \Gamma_{j'k'}^i = 2f_2^{-1} \frac{\partial f_2}{\partial z^j} \delta_k^i, \quad \Gamma_{j'k'}^{i'} = 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \delta_{k'}^{i'}, \quad \Gamma_{j'k'}^2 = \Gamma_{j'k'}^{2'} \quad 15$$

$$\Gamma_{jk'}^i = \Gamma_{j'k}^i = \Gamma_{jk}^{i'} = \Gamma_{j'k}^{i'} = 0 \quad 16$$

$$({}_{f_2}M_1 \times_{f_1} M_2, G)$$

$$T_{\gamma\alpha}^\beta$$

$$T_{jk}^i = T_{jk}^1, \quad T_{j'k'}^i = T_{j'k'}^{2'} \quad 17$$

$$T_{j'k}^i = 2f_2^{-1} \frac{\partial f_2}{\partial z^{j'}} \delta_k^i, \quad T_{jk'}^i = -2f_2^{-1} \frac{\partial f_2}{\partial z^{k'}} \delta_j^i \quad 18$$

$$T_{j'k}^{i'} = -2f_1^{-1} \frac{\partial f_1}{\partial z^k} \delta_{j'}^{i'}, \quad T_{jk'}^{i'} = 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \delta_{k'}^{i'} \quad 19$$

$$T_{j'k'}^i = T_{jk}^{i'} = 0 \quad 20$$

$$2 \quad \alpha = k, \beta = i, \gamma = j \quad 15$$

$$T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i = \Gamma_{jk}^1 - \Gamma_{kj}^1 = T_{jk}^1$$

$$({}_{f_1}M_1 \times_{f_1} M_2, G)$$

$$({}_{f_2}M_1 \times_{f_1} M_2, G)$$

$$(M_1, g)$$

$$(M_2, h)$$

$$f_1 \quad f_2$$

$$1 \quad ({}_{f_2}M_1 \times_{f_1} M_2, G)$$

$$T_{\gamma\alpha}^\beta = 0$$

$$1$$

$$\begin{cases} T_{jk}^1 = 0 \\ T_{j'k'}^{2'} = 0 \\ \frac{\partial f_1}{\partial z^k} = 0 \\ \frac{\partial f_2}{\partial z^{k'}} = 0 \end{cases} \quad 21$$

21

 $(M_1, g) \quad (M_2, h)$

21

 $f_1 \quad f_2$ $(_{f_1}M_1 \times_{f_1} M_2, G)$ $T_{\gamma\alpha\bar{\eta}}$

$$T_{jk\bar{l}} = f_2^2 T_{jk\bar{l}}^1, \quad T_{j'k'\bar{l}'} = f_1^2 T_{j'k'\bar{l}'}^2 \quad 22$$

$$T_{jk\bar{l}} = -2f_2 \frac{\partial f_2}{\partial z^{k'}} g_{j\bar{l}}, \quad T_{j'k'\bar{l}'} = 2f_1 \frac{\partial f_1}{\partial z^{j'}} h_{k'\bar{l}'} \quad 23$$

$$T_{j'k'\bar{l}'} = 2f_2 \frac{\partial f_2}{\partial z^{j'}} g_{k'\bar{l}'}, \quad T_{jk\bar{l}} = -2f_1 \frac{\partial f_1}{\partial z^k} h_{j\bar{l}} \quad 24$$

$$T_{jk\bar{l}} = T_{j'k'\bar{l}'} = 0 \quad 25$$

8 $\alpha = k, \gamma = j, \eta = l$

$$T_{jk\bar{l}} = G_{\beta\bar{i}} T_{jk}^\beta = G_{i\bar{l}} T_{jk}^i + G_{i'\bar{l}'} T_{jk}^{i'} \quad 26$$

17

13

26

$$T_{jk\bar{l}} = f_2^2 g_{i\bar{l}} T_{jk}^i = f_2^2 T_{jk\bar{l}}^1$$

 $(_{f_1}M_1 \times_{f_1} M_2, G)$ τ_γ

$$\tau_j = \tau_j^1 + 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \quad 27$$

$$\tau_{j'} = \tau_{j'}^2 + 2f_2^{-1} \frac{\partial f_2}{\partial z^{j'}} \quad 28$$

7 $\gamma = j$

$$\tau_j = G^{\bar{\eta}\alpha} T_{j\alpha\bar{\eta}} = G^{\bar{l}k} T_{jk\bar{l}} + G^{\bar{l}'k'} T_{j'k'\bar{l}'} + G^{\bar{l}k} T_{jk\bar{l}} + G^{\bar{l}'k'} T_{j'k'\bar{l}'} \quad 29$$

14

2

29

$$\begin{aligned} \tau_j &= f_2^{-2} g^{\bar{l}k} f_2^2 T_{jk\bar{l}}^1 + f_1^{-2} h^{\bar{l}'k'} 2f_1 \frac{\partial f_1}{\partial z^j} h_{k'\bar{l}'} \\ &= g^{\bar{l}k} T_{jk\bar{l}}^1 + 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \\ &= \tau_j^1 + 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \end{aligned}$$

28

 $(_{f_1}M_1 \times_{f_1} M_2, G)$ $(_{f_2}M_1 \times_{f_2} M_2, G)$

$$\begin{cases} \tau_j^1 + 2f_1^{-1} \frac{\partial f_1}{\partial z^j} = 0 \\ \tau_{j'}^2 + 2f_2^{-1} \frac{\partial f_2}{\partial z^{j'}} = 0 \end{cases} \quad 30$$

2 $(_{f_2}M_1 \times_{f_2} M_2, G)$ $\tau_\gamma = 0$

$$\begin{cases} \tau_j = 0 \\ \tau_{j'} = 0 \end{cases} \quad 31$$

27

28

31

30

 $(_{f_1}M_1 \times_{f_1} M_2, G)$ $f_1 \quad f_2$ $(_{f_2}M_1 \times_{f_2} M_2, G)$ $(M_1, g) \quad (M_2, h)$

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